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## Current fluctuations at a phase transition

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The ABC model is a simple diffusive one-dimensional non-equilibrium system which exhibits a phase transition. Here we show that the cumulants of the currents of particles through the system become singular near the phase transition. At the transition, they exhibit an anomalous dependence on the system size (an anomalous Fourier's law). An effective theory for the dynamics of the single mode which becomes unstable at the transition allows one to predict this anomalous scaling.

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A lot of work has been devoted recently to the study of the fluctuations of the current of heat or of particles through non-equilibrium one dimensional systems [1–12]. In such studies the basic quantity one considers is the total flux  $Q(t)$  of energy or of particles through a section of the system during time  $t$ . In the steady state this flux  $Q(t)$  fluctuates due to the randomness of the initial condition for purely deterministic models and due to the noisy dynamics in stochastic models (here we only discuss classical systems: see [13–16] for the quantum case). If one assumes that the total energy or the total number of particles in the system remains bounded, the average current  $\lim_{t \rightarrow \infty} \frac{\langle Q(t) \rangle}{t}$  as well as the higher cumulants  $\lim_{t \rightarrow \infty} \frac{\langle Q(t)^n \rangle_c}{t^n}$  of the flux  $Q(t)$  do not depend on the section of the system where this flux is measured.

For a one dimensional system of length  $L$ , a central question is the size dependence of these cumulants[17]. In particular one would like to know whether a given system satisfies Fourier's law, meaning that, for large  $L$ , the average current scales like  $1/L$ :

$$\lim_{t \rightarrow \infty} \frac{\langle Q(t) \rangle}{t} \simeq \frac{A_1}{L} \quad (1)$$

where the prefactor  $A_1$  depends on the temperatures  $T_1$  and  $T_2$  of the two heat baths or on the chemical potentials  $\mu_1$  and  $\mu_2$  of the two reservoirs of particles at the ends of the system. At equilibrium ( $T_1 = T_2$  or  $\mu_1 = \mu_2$ ) the prefactor  $A_1$  in (1) vanishes but the question of the validity of Fourier's law remains. One then wants to know whether the second cumulant of  $Q(t)$  scales like  $1/L$ .

$$\lim_{t \rightarrow \infty} \frac{\langle Q(t)^2 \rangle - \langle Q(t) \rangle^2}{t} = \lim_{t \rightarrow \infty} \frac{\langle Q(t)^2 \rangle_c}{t} \simeq \frac{A_2}{L}. \quad (2)$$

One can show that (2) holds for diffusive systems such as the SSEP (symmetric simple exclusion process)[1, 4, 18, 19] or the KMP (Kipnis-Marchioro-Presutti) model[20]. The macroscopic fluctuation theory developed by Bertini *et al.* [2, 21, 22] allows one also to determine[1] all the cumulants of the flux  $Q(t)$ , with the result that they all scale with system size as  $1/L$ .

$$\lim_{t \rightarrow \infty} \frac{\langle Q(t)^n \rangle_c}{t^n} \simeq \frac{A_n}{L} \quad (3)$$

Even corrections of order  $1/L^2$  have been computed in some cases[6, 7].

For mechanical systems with deterministic dynamics, in particular systems which conserve momentum, the average current scales as a non-integer power of the system size:

$$\lim_{t \rightarrow \infty} \frac{\langle Q(t) \rangle}{t} \simeq \frac{B_1}{L^{1-\alpha}}$$

The exponent  $\alpha$  takes the value  $1/2$  for some exactly soluble special models[3]. Values ranging from 0.25 to 0.4 have also been reported in simulations depending on the model considered[23–27]. Theoretical predictions based on a mode coupling approach[28, 29] or on renormalization group calculations[30] confirm this anomalous Fourier's law. Less is known on the size dependence of the higher cumulants, which are numerically harder to measure, except that they vary as power laws of the system size, with exponents which seem to depend on the geometry[31].

Here we consider the ABC model[32, 33], a diffusive system which is known to exhibit a phase transition[34–38]: we study the fluctuations of the current near this transition. Generically, outside the transition the cumulants have a diffusive scaling (3). Here, we show that the amplitudes  $A_n$  become singular as one approaches the transition, and that the cumulants of  $Q(t)$  exhibit anomalous scalings at the transition. When the transition is second order, due to the destabilization of a single Fourier mode of the density[34], the fluctuations in the whole critical regime can be understood in terms of a Langevin equation for a single complex variable which represents the amplitude and the phase of this Fourier mode.

### I. DEFINITION

The ABC model[32–38] is a one-dimensional lattice gas, where each site is occupied by one of three types of particles,  $A$ ,  $B$  and  $C$ . Neighboring sites exchange particles at the rates

$$\begin{aligned} AB &\xrightleftharpoons{q} BA \\ BC &\xrightleftharpoons{q} CB \\ CA &\xrightleftharpoons{q} AC \end{aligned}$$

with an asymmetry  $q \leq 1$ . Here, we consider the model on a ring of  $L$  sites. Since the rates are invariant under cyclic permutations of  $\{A, B, C\}$ , most of the equations below will be written for a species  $a \in \{A, B, C\}$ , with  $b$  and  $c$  denoting respectively the next and previous species. When  $q$  scales as

$$q = e^{-\beta/L},$$

the dynamics become diffusive: in particular, the probability that site  $1 \leq k \leq L$  is occupied by a particle of type  $a$  behaves as

$$\text{Pro}[s_k(t) = a] \simeq \rho_a(k/L, t/L^2),$$

where the macroscopic density profiles  $\rho_a(x, \tau)$  follow a local Fourier's law: if  $j_a(x, \tau)$  is the current associated to  $\rho_a$ , then

$$j_a = -\partial_x \rho_a + \beta \rho_a (\rho_c - \rho_b), \quad (4)$$

which, together with the conservation law

$$\partial_\tau \rho_a = -\partial_x j_a, \quad (5)$$

gives the hydrodynamic equations[34] satisfied by the density:

$$\partial_\tau \rho_a = \partial_x^2 \rho_a + \beta \partial_x \rho_a (\rho_b - \rho_c). \quad (6)$$

These equations conserve the fact that  $\sum_a \rho_a(x, \tau) = 1$  (each site is occupied by one of the three species) and  $\int_0^1 dx \rho_a(x, \tau) = r_a$ , where  $r_a$  is the total density of particles of type  $a$ . The deterministic equations (4) and (6) are only valid in the large  $L$  limit, for diffusive time scales, i.e.  $t \sim L^2$ .

For small  $\beta$ , the constant density profiles  $\rho_a(x) = r_a$  are a stable stationary solution of (6). These constant profiles become linearly unstable above a critical value  $\beta_*$ [34] given by

$$\beta_* = \frac{2\pi}{\sqrt{\Delta}} \quad \text{with} \quad \Delta = 1 - 2 \sum_a r_a^2, \quad (7)$$

so that the long-time limit of (6) becomes a function  $\bar{\rho}_a(x)$  of the space variable  $x$  for  $\beta > \beta_*$ . It has been argued[39] (and checked numerically) that, in the steady state, these modulated profiles do not move. If the phase transition to the modulated phase is second order, then it should occur at  $\beta = \beta_*$  given by (7). A first-order transition may however occur at  $\beta < \beta_*$ : this should certainly[34] be the case at least for  $\Lambda < 0$ , with

$$\Lambda = \sum_a r_a^2 - 2 \sum_a r_a^3. \quad (8)$$

In the following, we study the integrated current  $Q_A(t)$  of  $A$  particles through the system during time  $t$ :

$$\begin{aligned} Q_A(t) &= \frac{1}{L} \int_0^t dt' \int_0^1 dx j_A(x, t') \\ &= L \int_0^{t/L^2} d\tau \int_0^1 dx j_A(x, \tau) d\tau. \end{aligned} \quad (9)$$

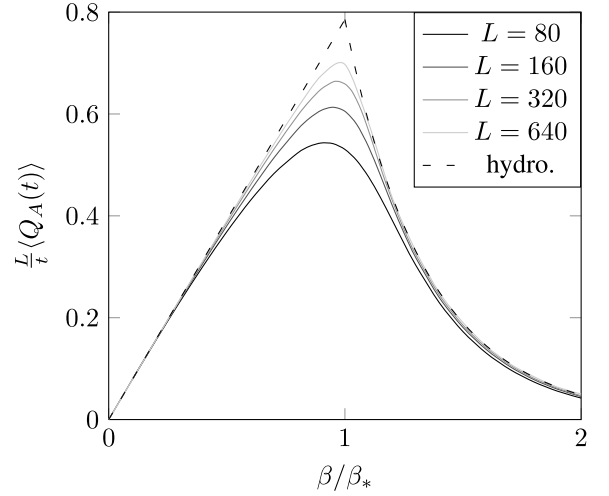


FIG. 1: First cumulant of the integrated current of  $A$  particles in the  $ABC$  model with  $r_A = r_B = 1/4$ , for  $0 \leq \beta \leq 2\beta_*$ . Measurements of  $\langle Q_A \rangle$  in Monte-Carlo simulations of systems of 80 to 640 sites are compared to the prediction of the hydrodynamic equation (11).

This space average fluctuates with time, and we will be interested in its cumulants, as in (3). Because the difference between the space average and the flux through a section remains bounded, the cumulants of the flux through any section are the same as those of this space average in the long time limit[39].

## II. MEAN CURRENT

As the steady-state profiles  $\bar{\rho}_a(x)$  are time independent[39], (5) implies that the steady-state currents are constant,  $\bar{j}_a(x) = J_a$ . They are given by

$$J_a = \beta \int_0^1 dx \bar{\rho}_a(x) (\bar{\rho}_c(x) - \bar{\rho}_b(x)). \quad (10)$$

Then, from (9),

$$\frac{\langle Q_A(t) \rangle}{t} \sim \frac{\beta}{L} \int_0^1 dx \bar{\rho}_A(x) (\bar{\rho}_C(x) - \bar{\rho}_B(x)). \quad (11)$$

$\langle Q_A(t) \rangle$  can thus be obtained by calculating numerically the long-time limit of (6) and then integrating (10). In Figure 1, we compare the results of this calculation to numerical measurements obtained by simulating finite systems of 80 to 640 sites, for  $r_A = r_B = 1/4$  and  $0 \leq \beta \leq 2\beta_*$ , with  $\beta_*$  given by (7).

For  $\beta \leq \beta_*$ , the stability of the flat density profiles  $\bar{\rho}_a(x) = r_a$  leads to a current  $\langle Q_A(t) \rangle \sim \frac{1}{L} \beta r_A (r_C - r_B)$ ; on the other hand, the dependence in  $\beta$  becomes non-trivial for  $\beta > \beta_*$ , with a cusp at  $\beta_*$ .

For  $\beta \downarrow \beta_*$ , the steady-state profiles are known (see [34] or (27) and (28) below) to take the form

$$\bar{\rho}_a(x) = r_a + \sqrt{\beta - \beta_*} (K_a e^{2i\pi x} + cc.) + \mathcal{O}(\beta - \beta_*) \quad (12)$$

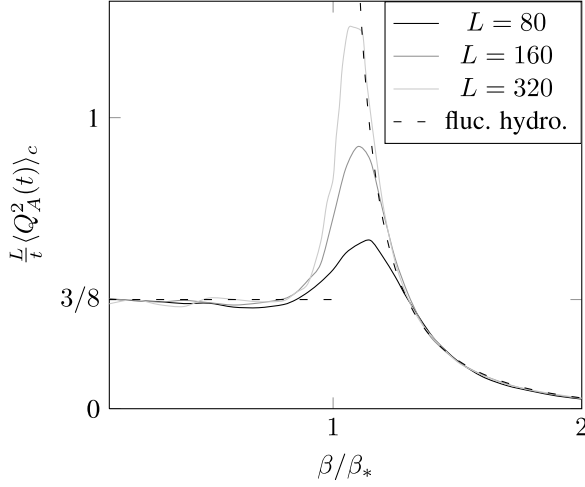


FIG. 2: Second cumulant of the integrated current of  $A$  particles in the  $ABC$  model with  $r_A = r_B = 1/4$ , for  $0 \leq \beta \leq 2\beta_*$ . Measurements of  $\langle Q_A^2 \rangle_c$  in systems with 80 to 320 sites are compared to those obtained from the macroscopic fluctuation theory (13,14).

with known constants  $K_A$ ,  $K_B$  and  $K_C$ , leading to an analytic expression for  $\langle Q_A(t) \rangle$  around  $\beta_*^+$ :

$$\frac{L}{t} \langle Q_A(t) \rangle_{\beta \downarrow \beta_*} \simeq (r_C - r_B) \left[ \beta r_A - \frac{\Delta^2}{\Lambda} (\beta - \beta_*) \right].$$

### III. FLUCTUATION THEORY

The hydrodynamic equations (6) describe the deterministic evolution of the density profiles  $\rho_a(x, \tau)$  in the large  $L$  limit. For a large, but finite system, one has to take into account stochastic corrections. This can be done using fluctuating hydrodynamics, where the expression (4) of the current is replaced with [4, 19]

$$j_a = q_a + \frac{1}{\sqrt{L}} \eta_a(x, \tau) \quad (13)$$

where  $q_a = -\partial_x \rho_a + \beta \rho_a (\rho_c - \rho_b)$  is the right-hand side of (4), and where the  $\eta_a$  are Gaussian noises such that  $\sum_a \eta_a = 0$  and

$$\langle \eta_a(x, \tau) \eta_{a'}(x', \tau') \rangle = \sigma_{aa'}(x, \tau) \delta(x - x') \delta(\tau - \tau'),$$

with  $\sigma_{aa} = 2\rho_a(1 - \rho_a)$  and  $\sigma_{aa'} = -2\rho_a\rho_{a'}$  for  $a \neq a'$ . Alternatively, the macroscopic fluctuation theory [40] expresses (13) as a large deviation principle, with the probability of observing time-dependent density profiles  $\rho_a(x, \tau)$  given by

$$\text{Pro}[\rho_a(x, \tau)] \propto \exp \left[ -L \iint dx d\tau \sum_a \frac{(j_a - q_a)^2}{4\rho_a} \right].$$

From this formulation, the generating function of the current  $Q_A(t)$  is the solution of the optimization problem

$$\log \langle e^{\lambda Q_A(t)} \rangle = \max_{\rho_a, j_a} L \int_0^{t/L^2} d\tau \int dx \left[ \lambda j_A - \sum_a \frac{(j_a - q_a)^2}{4\rho_a} \right]. \quad (14)$$

Finding the density and current profiles  $\rho_a(x, \tau)$  and  $j_a(x, \tau)$  which maximize (14) is not an easy task. We assume that, for large  $t$ , the optimum in (14) is achieved by profiles of fixed shape which may drift with a constant velocity  $v$ . In order to obtain  $\langle Q_A^2(t) \rangle_c$ , we compute  $\log \langle e^{\lambda Q_A(t)} \rangle$  to order 2 in  $(\lambda, v)$ , before optimizing over  $v$ : one then gets optimization equations satisfied by these moving profiles, which we solved numerically in the case  $r_A = r_B = 1/4$  for  $0 \leq \beta \leq 2\beta_*$ .

For  $\beta < \beta_*$ , the constant profiles  $\bar{\rho}_a(x) = r_a$  are still optimal for  $\lambda \neq 0$ , yielding

$$\langle Q_A^2(t) \rangle_c \simeq \frac{2t}{L} r_A (1 - r_A).$$

For  $\beta > \beta_*$ , one has to take into account the dependence of the optimal profiles in  $v$  and  $\lambda$ . This leads (as will be shown in the longer [41]) to a second cumulant which diverges at  $\beta = \beta_*$ . This divergence can be computed exactly thanks to the knowledge of the steady-state profiles (12) for  $\beta \downarrow \beta_*$ , leading to

$$\langle Q_A^2(t) \rangle_c \underset{\beta \downarrow \beta_*}{\simeq} \frac{t}{L} \frac{12\pi r_A r_B r_C (r_B - r_C)^2}{\Lambda \sqrt{\Delta} (\beta - \beta_*)}.$$

### IV. CRITICAL REGIME FOR THE DETERMINISTIC HYDRODYNAMICS

In this section, we analyse how the deterministic equations (6), exact in the  $L \rightarrow \infty$  limit, behave in the neighborhood of  $\beta_*$ . The stability analysis of the constant profiles  $\rho_a = r_a$  shows that, as  $\beta$  crosses  $\beta_*$ , only the first Fourier modes of the  $\rho_a$  become unstable. For  $\beta$  close to  $\beta_*$ , one therefore expects this mode to relax more slowly than the other Fourier modes.

One can write, from (6), an evolution equation for these slow modes when  $\beta$  is close to  $\beta_*$ , and show that they decay as a power law (instead of an exponential) in the critical regime. To do so, we separate in  $\rho_a(x, \tau)$  the first Fourier mode,  $R_a(\tau)e^{2i\pi x}$ , from the other modes,  $\tilde{\rho}_a(x, \tau)$ , so that

$$\rho_a(x, \tau) = r_a + R_a(\tau)e^{2i\pi x} + cc. + \tilde{\rho}_a(x, \tau), \quad (15)$$

$$j_a(x, \tau) = J_a(\tau) + \frac{i}{2\pi} \dot{R}_a e^{2i\pi x} + cc. - \partial_\tau \int_0^x dy \tilde{\rho}_a(y, \tau)$$

with  $\tilde{\rho}_a \ll R_a \ll r_a$ . We also suppose that  $\rho_a(x, \tau)$  varies slowly, so that  $\partial_\tau R_a \equiv \dot{R}_a \ll R_a$  and  $\partial_\tau \tilde{\rho}_a \ll \tilde{\rho}_a$ ; finally, we set

$$\beta = \beta_*(1 + \gamma)$$

with  $\gamma \ll 1$ . The leading order of (4) then becomes, when

projected on the first and second Fourier modes,

$$2i\pi R_a = \beta_* [R_a(r_c - r_b) + r_a(R_c - R_b)] \quad (16)$$

$$\begin{aligned} \partial_x \tilde{\rho}_a &= \beta_* [\tilde{\rho}_a(r_c - r_b) + r_a(\tilde{\rho}_c - \tilde{\rho}_b) \\ &\quad + R_a(R_c - R_b)e^{4i\pi x} + cc.] \end{aligned} \quad (17)$$

Equation (16) relates the leading orders of  $R_B$  and  $R_C$  to  $R_A$ , so that

$$\begin{cases} R_B(\tau) = \frac{2r_C - 1 - i\sqrt{\Delta}}{2r_A} R_A(\tau) + x_B(\tau) \\ R_C(\tau) = \frac{2r_B - 1 + i\sqrt{\Delta}}{2r_A} R_A(\tau) + x_C(\tau) \end{cases} \quad (18)$$

with  $x_B, x_C \ll R_A$ . Equation (17) shows that, at leading order,  $\tilde{\rho}_a$  is of the form

$$\tilde{\rho}_a(x, \tau) = \varphi_a e^{4i\pi x} + cc. \quad (19)$$

with

$$4i\pi\varphi_a = \beta_* [\varphi_a(r_c - r_b) + r_a(\varphi_c - \varphi_b) + R_a(R_c - R_b)],$$

whose solution is

$$\varphi_a = \frac{1 - 2r_a}{\Delta} R_a^2. \quad (20)$$

Then, the next-to-leading order of the first Fourier mode of (4) reads

$$\begin{aligned} \frac{i}{2\pi} \dot{R}_a &= 2i\pi\gamma R_a + \beta_* [(r_c - r_b - i\sqrt{\Delta})x_a + r_a(x_c - x_b) \\ &\quad + \varphi_a(R_c^* - R_b^*) + R_a^*(\varphi_c - \varphi_b)] \end{aligned} \quad (21)$$

(with  $x_A \equiv 0$ ). The  $x_a$  can be eliminated by multiplying the equation over  $\dot{R}_A$  by  $(r_C - r_B + i\sqrt{\Delta})$ , the one over  $\dot{R}_C$  by  $r_A$ , the one over  $\dot{R}_B$  by  $-r_A$ , and by summing, which yields

$$\dot{R}_A = 4\pi^2 \left( \gamma - \frac{2\Lambda}{\Delta^2} \frac{|R_A|^2}{r_A} \right) R_A, \quad (22)$$

with  $\Lambda$  given by (8).

For  $\beta = \beta_*$  ( $\gamma = 0$ ), the first Fourier mode  $R_A$  decays as a power law instead of an exponential:

$$R_A(\tau) = \frac{R_A(0)}{\sqrt{1 + \frac{16\pi^2\Lambda}{r_A\Delta^2} |R_A(0)|^2 \tau}},$$

with an amplitude which does not depend on the initial condition for large  $\tau$ .

## V. CRITICAL BEHAVIOR OF THE MFT

We now return to the noisy equation (13) and try to obtain a noisy version of (22). By analogy with the deterministic case above, we suppose that the first Fourier mode  $R_a$  of  $\rho_a$  varies more slowly, but with a larger amplitude than the other modes, so that  $\dot{R}_a \ll R_a$  and  $\tilde{\rho}_a \ll R_a$  in (15). When replacing (4) with (13), the leading-order equation (16) is not modified,

so that (18) still holds : however, the next-to-leading order equation (21) is replaced with

$$\begin{aligned} \frac{i}{2\pi} \dot{R}_a &= 2i\pi\gamma R_a + \beta_* [(r_c - r_b - i\sqrt{\Delta})x_a + r_a(x_c - x_b) \\ &\quad + \varphi_a(R_c^* - R_b^*) + R_a^*(\varphi_c - \varphi_b)] + \frac{\nu_a(\tau)}{\sqrt{L}}, \end{aligned} \quad (23)$$

where the  $\nu_a$  are projections of the  $\eta_a(x, \tau)$  on the first Fourier mode:

$$\nu_a(\tau) = \int_0^1 dx e^{-2i\pi x} \eta_a(x, \tau),$$

so that  $\langle \nu_a(\tau) \nu_{a'}^*(\tau') \rangle = \sigma_{aa'} \delta(\tau - \tau')$  and  $\langle \nu_a(\tau) \nu_{a'}(\tau') \rangle = 0$ . The second Fourier modes  $\varphi_a$  (19) satisfy the equations

$$\begin{aligned} \frac{i}{4\pi} \dot{\varphi}_a &= -4i\pi\varphi_a + \beta_* [\varphi_a(r_c - r_b) + r_a(\varphi_c - \varphi_b) \\ &\quad + R_a(R_c - R_b)] + \frac{\nu_a^{(2)}(\tau)}{\sqrt{L}}, \end{aligned}$$

where the  $\nu_a^{(2)}$  are projections of the  $\eta_a(x, \tau)$  as well : hence, they fluctuate around their non-noisy expression (20). In (23), however, these fluctuations (of amplitude  $1/\sqrt{L}$ ) are multiplied by  $R_a^*$ , so that they are of smaller amplitude than the noisy term  $\nu_a(\tau)/\sqrt{L}$  : therefore, the  $\varphi_a$  can be replaced by their expression (20) in (23).

Taking a linear combination of (23) to eliminate the  $x_a$  as in the deterministic case (21), we then obtain

$$\dot{R}_A = 4\pi^2 \left( \gamma - \frac{2\Lambda}{\Delta^2} \frac{|R_A|^2}{r_A} \right) R_A + \frac{\mu_A(\tau)}{\sqrt{L}},$$

with  $\mu_A$  a linear combination of the  $\nu_a$ , which verifies

$$\langle \mu_A(\tau) \mu_A^*(\tau') \rangle = \frac{24\pi^2 r_A^2 r_B r_C}{\Delta} \delta(\tau - \tau').$$

Finally, the change of variables

$$\begin{aligned} R_A(\tau) &= \sqrt[4]{\frac{3\Delta r_A^3 r_B r_C}{\Lambda L}} f(\bar{\tau}) \\ &\quad \text{with } \bar{\tau} = 8\pi^2 \frac{\sqrt{3\Lambda r_A r_B r_C}}{\Delta^{3/2}} \frac{\tau}{\sqrt{L}} \end{aligned} \quad (24)$$

leads to a simple rescaled equation:

$$\begin{aligned} \frac{df}{d\bar{\tau}} &= (\bar{\gamma} - |f(\bar{\tau})|^2) f(\bar{\tau}) + \mu(\bar{\tau}) \\ &\quad \text{with } \bar{\gamma} = \sqrt{L} \frac{\Delta^{3/2}}{2\sqrt{3\Lambda r_A r_B r_C}} \frac{\beta - \beta_*}{\beta_*} \end{aligned} \quad (25)$$

and with  $\mu$  such that  $\langle \mu(\bar{\tau}) \mu^*(\bar{\tau}') \rangle = \delta(\bar{\tau} - \bar{\tau}')$ .

Therefore, a system of size  $L$  exhibits a critical regime  $|\beta - \beta_*| \sim 1/\sqrt{L}$  in which the density profiles  $\rho_a(x, \tau)$  fluctuate as sine waves of period 1, with an amplitude scaling as  $1/L^{1/4}$  on a time scale of order  $1/\sqrt{L}$ . The rescaled fluctuations,

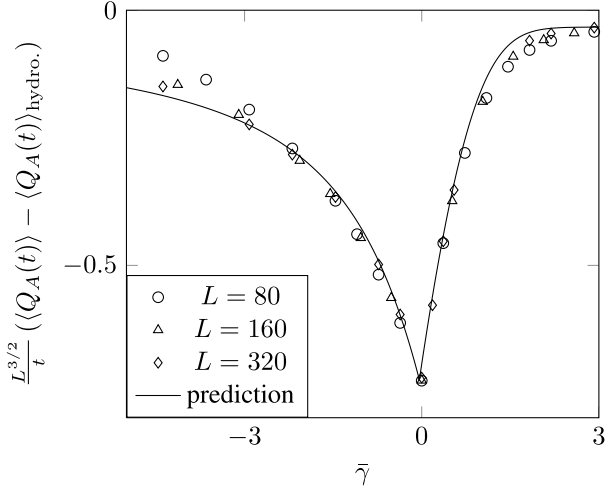


FIG. 3: Rescaled deviation of the first cumulant of  $A$  particles,  $\langle Q_A(t) \rangle$ , from its deterministic hydrodynamics prediction (11), as a function of  $\bar{\gamma}$  (25), for  $r_A = r_B = 1/4$ . Numerical measurements for systems of 80 to 320 sites are compared with our analysis of the critical point (27,28).

$f(\bar{\tau})$ , follow a (complex) damped Langevin dynamics in the quartic potential

$$V(f) = -\bar{\gamma} \frac{|f|^2}{2} + \frac{|f|^4}{4},$$

and the probability distribution of  $f(\bar{\tau})$ ,  $P(r, \theta, \bar{\tau}) = \text{Pro}[f(\bar{\tau}) \simeq r e^{i\theta}]$ , satisfies the Fokker-Planck equation

$$\partial_{\bar{\tau}} P = \frac{1}{r} \partial_r \left[ (r^2 - \bar{\gamma}) r^2 P + \frac{1}{4} r \partial_r P \right] + \frac{1}{4r^2} \partial_{\theta}^2 P. \quad (26)$$

## VI. CRITICAL FLUCTUATIONS OF THE CURRENT

Let us now discuss the consequences of the slow fluctuations of the first Fourier mode of the density described above on the integrated particle current of  $a$  particles,  $Q_a(t)$ . From (13) and (15), we can express the average instantaneous current,  $J_a$ , in terms of  $f(\bar{\tau})$ :

$$J_a(\tau) = \beta r_a (r_b - r_c) + 2\beta (r_b - r_c) \sqrt{\frac{3\Delta r_a r_b r_c}{\Lambda L}} |f(\bar{\tau})|^2 + \frac{1}{\sqrt{L}} G_a(\tau) + \mathcal{O}\left(\frac{1}{L}\right)$$

where  $G_a$ , the space average of the noise  $\eta_a$ , is such that

$$\langle G_a(\tau) G_a(\tau') \rangle = 2r_a(1 - r_a) \delta(\tau - \tau') + \mathcal{O}(1/\sqrt{L}).$$

Therefore, the contributions of the fluctuations of the first Fourier mode  $f$  and of the noise  $G_a$  to  $J_a$  are of comparable amplitude. The fluctuations of  $f$ , however, occur on the

slower time scale  $\bar{\tau} \sim \tau/\sqrt{L}$ : hence, they become dominant in the integrated current (9)

$$Q_a(t) = L \int_0^{t/L^2} J_a(\tau) d\tau,$$

with the  $n$ -points time correlation function of  $f(\bar{\tau})$  giving rise to an anomalous growth of the  $n$ -th cumulant of  $Q_a$ ,  $\langle Q_a^n(t) \rangle_c$ . More precisely, we find that

$$\langle Q_a(t) \rangle \simeq \frac{t}{L} \beta r_a (r_c - r_b) + \frac{2t}{L^{3/2}} \beta (r_b - r_c) \sqrt{\frac{3\Delta r_a r_b r_c}{\Lambda}} C_1(\bar{\gamma}) \quad (27)$$

and

$$\langle Q_a^n(t) \rangle_c \simeq \frac{t}{L^{5/2-n}} \frac{8\pi^2 \sqrt{3\Delta r_a r_b r_c}}{\Delta^{3/2}} \left[ \frac{\Delta^{3/2} (r_b - r_c)}{2\pi\Lambda} \right]^n C_n(\bar{\gamma})$$

with  $\bar{\gamma}$  as defined in (25) and with

$$C_n(\bar{\gamma}) = \lim_{\bar{\tau} \rightarrow \infty} \frac{1}{\bar{\tau}} \int_0^{\bar{\tau}} d\bar{\tau}_1 \dots d\bar{\tau}_n \langle |f(\bar{\tau}_1) \dots f(\bar{\tau}_n)|^2 \rangle_c$$

a time integral of the  $n$ -point correlation function of  $f(\bar{\tau})$ .

Because  $f(\bar{\tau})$  follows a Langevin equation in a quartic potential (25), one does not have simple analytical expressions for the  $C_n(\bar{\gamma})$  for  $n \geq 2$ .  $C_1(\bar{\gamma})$ , on the other hand, only depends on the stationary average of  $|f(\bar{\tau})|^2$ : since the stationary state of the Fokker-Planck equation (26) is  $P(r, \theta) \propto e^{2\bar{\gamma}r^2 - r^4}$ , we obtain easily that

$$C_1(\bar{\gamma}) = \bar{\gamma} + \frac{1}{2} \frac{e^{-\bar{\gamma}^2}}{\int_{-\infty}^{\bar{\gamma}} e^{-z^2} dz}. \quad (28)$$

Thus, we obtain an analytic expression for the deviation of  $\langle Q_A(t) \rangle$  from its hydrodynamics prediction (11) in the neighborhood of  $\beta_*$ . In Figure 3, we compare this expression to numerical simulations of systems of 80 to 320 sites, for  $r_A = r_B = 1/4$ .

## VII. CONCLUSION

In this letter we have seen that a driven diffusive system like the ABC model, at a second order phase transition, may exhibit anomalous Fourier's law at least for the second and higher cumulants. This is reminiscent of the cumulants of the current which diverge with the system size in the TASEP (the totally asymmetric exclusion process) along the first order transition line[12]. The mechanism is however different. Here the large fluctuations can be understood by analysing the dynamics of the first Fourier mode which becomes unstable at the transition whereas, in the TASEP, the large fluctuations of the current are due to the presence of the shock.

The anomalous current fluctuations of the ABC model at the phase transition are accompanied by anomalous long range density fluctuations which can also be understood in terms of the slow noisy dynamics of the first Fourier mode



(the density fluctuations will be discussed in the forthcoming longer version of the present letter[41]).

An interesting open question would be to compare the anomalous density and the current fluctuations of the ABC model at the transition with those of momentum conserv-

ing mechanical models, in particular through the dynamics of their slow modes. Another interesting question would be to study the current fluctuations through other lattice gases (such as an Ising model) when there is coexistence of several phases at equilibrium.

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